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How secondary level teachers and students impose personal structure on fractional expressions and equations—an expert-novice study

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Abstract While an algebraic expression is typically assigned a regular structure, this article introduces the concept of personal structure (*Strukturierung*); here, the structuring of an algebraic expression is understood as the act of forming relationships between its parts. This concept is used for the analysis of interviews in which experts and novices talk about the personal structures they produced when looking at fractional expressions and equations. The results show, firstly, that experts will structure a given fractional expression in various different ways. Secondly, four categories have been identified: structuring can mean that an expression is made more visually straightforward, changed, reinterpreted or classified. This is meaningful for negotiating the appropriateness of the personal structures in teaching algebra.

Keywords Fractional expressions and equations · Personal structures · Expert-novice

1 Introduction

As Kieran (1989) has pointed out, the recognition and use of structures is the core of high school algebra. The significance of this statement is highlighted by many studies on how pupils structure given terms and equations (Hoch & Dreyfus, 2004; Kirshner & Awtry, 2004; Linchevski & Livneh, 1999; Malle, 1993; Mayer, 1982; Sfard & Linchevski, 1994; Star & Rittle-Johnson, 2008; Sweller & Cooper, 1985).

Typically, the structure of an algebraic expression is understood to be an objective mathematical property of the expression, which is useful for manipulating the expression. The concept of structure used in Kieran (1989) provides an illustration of this. She was one of the first to define the concept of the structure of an algebraic expression. Alongside the operations and relations which are represented (*surface structure*), she also stressed the concept of the properties of the operations which are represented (*systemic structure*). The term surface structure describes the relationships between the partial terms of the term in accordance with the hierarchy of operations, as well as the equality of two terms on the left

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and right of an equals sign. The systemic structure comprises all equivalent forms into which a term can be transformed in accordance with mathematical laws, or rather, all equations that are equivalent to an equation. “Thus the structure of the expression $3(x+2)+5$ includes the surface structure, [...] that is, the multiplication of 3 by $x+2$, followed by the addition of 5, along with the systemic structure, that is, the equivalent forms of the expression according to the properties of the given operations” (ibid., p. 34). So here the structure of an algebraic expression encompasses on the one hand its basic construction, and on the other its transformations. Malle (1993) understood the concept of structure differently. When Malle talks of “recognising term structures” (ibid., p. 190), he means the identification of partial terms. For example, according to Malle, the left side of the equation $4 \cdot x + 3 = 11$ can be structured in three ways: $\boxed{4 \cdot x + 3} = 11$, $\boxed{4 \cdot x} + \boxed{3} = 11$ or $\boxed{4} \cdot \boxed{x} + \boxed{3} = 11$ (ibid., p. 189). These term structures provide the division of the left side of the equation $4 \cdot \mathrm{x} + 3 = 11$ into individual units. They are determined by the principles that regulate the construction of a term. Therefore, structuring is for Malle the division of the expression into individual partial terms, whereby the possible divisions are provided only through the construction of the expression. More recently, others, most notably Hoch and Dreyfus (2004, 2010), have examined the structure of algebraic expressions. In expressions such as $30x^2 - 28x + 6$ and $(5x - 3)(6x - 2)$, they speak of the $ax^2 + bx + c$ structure. The varying representations, the “quadratic expression” and the “product of two linear factors” are, for Hoch and Dreyfus, two varying *interpretations* of the $ax^2 + bx + c$ structure. The structure recognised here is the normal form of quadratic expressions. Thus, for Hoch and Dreyfus, the structure of an algebraic equation denotes a class of problems that can be solved in a similar way. The class of quadratic equations serves as a particular example. This becomes particularly evident in Hoch and Dreyfus (2010) where the authors suggest five structures that are central to secondary education, namely $a^2 - b^2$, $a^2 + 2ab + b^2$, $ab + ac + ad$, $ax + b = 0$ and $ax^2 + bx + c = 0$. To summarise, the three approaches outlined above suggest that the structure of an algebraic expression should refer to its basic construction and its transformations, to its division into partial terms, or to the class of expressions to which it belongs mathematically. Structure is, according to these, a mathematical property of the expression.

There are no personal, or indeed incorrect structures, in the above conceptions. This is because the structures outlined above are oriented towards mathematical theory, mathematical definitions and theorems. But for reconstructing individual learning and thought processes, it is more appropriate to use terms which describe mathematical practice, i.e. mathematical activity. In this context personal structures, the interpretations of an expression which are formed by an individual, become important. This is why this article puts forward a definition of ‘personal structure’ with which to describe this kind of personal interpretation of an algebraic expression. The simplification of the term $\mathrm{x}(\mathrm{x}+1)-(\mathrm{x}+1)$ serves as an example of this. At first sight, this is a meaningless string of symbols. A person looks at this string of symbols, interprets it, and thus gives it meaning. This meaning depends on the person’s individual approach. A few possible interpretations follow. In the term, there are two identical parts $(x+1)$ and in the middle there is a minus sign; the term is mostly made up of brackets; in the term there is an x times $(x+1)$; there is an algebraic minus sign in the term; the term is a difference; the term is distributive. In this article, personal interpretations such as this are considered as ‘personal structures’, because manipulations arise from them in the same way that manipulations arise from the structures above.

Here, the ‘structuring’ of an algebraic expression is equated to relationships being formed between its parts. The ‘personal structure’ of an algebraic expression is, then, the product of this activity. This concept allows individual processes, *how* people recognise potential manipulations, to be described. What is the first thing they notice? Whereabouts in the expression do they look? What are they thinking when they do so? Which parts do they relate to each other? What do they deduce from this? Such questions form the core of this article. They focus on the actions that a person completes before performing any manipulations. Structuring is, then, a process whereby a person works out his/her own personal structure. Understanding how a person comes to manipulate the expression in that precise way is especially important because effective support and encouragement of learning begins at the level of individual thought processes. Teachers need to understand the personal structures created by the students. A teacher must be able to describe even inadequate personal structures that can lead to incorrect manipulation so that he/she can recognise that the student created what he/she believed to be the most reasonable structure. Only then will the teacher understand what meanings the student attributed to the string of symbols and how he/she, as a teacher, can provide a comprehensible answer. The purpose of teachers being able to describe students' personal structures is not to enable them to eradicate the students' inappropriate personal structures more easily. Rather, teachers should create relationships between inappropriate and appropriate personal structures in the lesson. This idea of teaching is subject to, for example, the research concept of educational reconstruction (van Dijk & Kattmann, 2007), the design of didactic structure in the sense of Lijnse (1995) or the didactic model of dialogic learning (Ruf & Gallin, 1998).

The first part of this article therefore provides a theoretical description of the idea of personal structures of algebraic expressions. There will be a particular focus on the meaning-forming character of personal structures. The second part introduces a study of the empirical formation of personal structures. In this there is an emphasis on the range of personal structures, whereas Rüede (2012) discusses connections between the four empirically identified categories of personal structures.

1.1 Structuring as a construction of meaning

There is fundamentally an internal and an external mathematical use of terms and equations. Kieran (2006) uses the terms *internal* and *external semantics*. So there are two ways that students can construct the meaning of algebraic expressions. In the first case, they use expressions for handling mathematical problems. For example, they structure expressions, manipulate them or depict them as a table of values or a graph. In the second case, the expressions relate to events and situations outside mathematics, and they model material problems. There is no doubt that the internal and external semantics are interdependent, especially when learning algebra. Nevertheless, this article focuses on the former and, within this, on structuring. This is because in the study that follows, the subjects were presented with algebraic expressions which were removed from any context outside of mathematics. The issue at hand is therefore how the subjects construct internal meaning in the algebraic calculation during their manipulation of expressions, and not how they construct external meaning when modelling real situations.

What now needs to be explained is how a person constructs this internal meaning. Because the argument that follows is based on a pragmatic concept of meaning, it is asserted in the first step below that structuring is indeed an action, while a second step sets out that the personal meaning of a personal structure does encompass actions such as these.

1.1.1 First step: structuring is an action

A person who creates relationships is performing an action. That is the fundamental reason for equating *structuring* with the act of forming relationships between parts of an expression in this article. These actions can manifest themselves in various ways. Three examples are given:

- The person explains what he/she is currently looking at, and which parts of the expression he/she is relating to other parts. Is he/she comparing the left side of an equation to the right side? Is he/she looking horizontally, vertically, or diagonally? Is he/she talking about coefficients or about what comes before and after? Through such statements the person expresses, amongst other things, the *geometry* (Fischer, 1984) of the string of algebraic symbols.
- The person learns how someone structures the expression differently, and articulates this structuring with his/her own words and gestures.
- The person points to individual parts of the expression, colours in individual parts on their print-out, draws arrows between individual parts or simply adds brackets for clarification purposes.

These are all actions. Three aspects are important here. Firstly, these actions are observable. Secondly, it follows that structuring can be empirically assessed. And thirdly, a *personal structure* can now be defined as the result of structuring actions such as these. A personal structure indicates *which* parts of the expression are related to each other and *how* they are related. In the empirical section that follows, many different personal structures are described. Therefore, the simplification of $\frac{2x}{2x-2} - 2 \cdot \frac{x}{2x-2}$ should suffice as an example here. We could imagine, for example, a first person who divides the expression up into two parts, one on each side of the minus sign, and treats it subsequently as a difference. In this case the person relates $\frac{2x}{2x-2}$ and $2 \cdot \frac{x}{2x-2}$ to each other in that they treat the first part as a minuend and the second as the subtrahend of a subtraction. This would be the personal structure. Perhaps the person then even tries to make the two parts equal each other, in order to arrive at an expression in the form of $A - A$. We could also imagine a second person who divides the two-dimensional arrangement into parts above and below the fraction line (except for the coefficient 2), relates the two denominators to each other and recognises that they are the same. A possible personal structure could consist of the person treating each of the parts below the fraction line as the common denominator of two fractions which are to be subtracted. A consequence would be that the person then takes the coefficient 2 up to the x and mentally extends the fraction line. Finally, a third person could focus on the first fraction and divide this up into one part above ($2x$) and two parts below ($2x$ and 2). These three parts would then appear almost like the three corners of a triangle. By relating these three parts to each other, the person perhaps recognises the common factor 2. This would be the third personal structure.

1.1.2 Second step: the personal meaning of a personal structure

The *personal meaning* of a personal structure of an expression is equated to the personal use of the personal structure, in line with a pragmatic concept of meaning. This use encompasses the *When* and the *How* (Sfard, 2008) of the personal structure. Therefore the personal meaning of a personal structure is reconstructed by demonstrating when (in which situations) the person has related the parts of an algebraic expression to one another and how. The

How of a personal structure is provided through the description of that personal structure, because this states how and which of these parts of the expression relate to one another. The When of a personal structure, however, is two-fold. Firstly, it is described through the 'lead-up' to the personal structure. This provides answers as to why it was structured in this particular way. How has the person structured similar expressions up till now? What did the person pay attention to in the expression? What reasons did he/she have for the personal structure? What were the basic conditions? The second part of the When of a personal structure lies in its assessment. This shows itself (often tacitly) in how the personal structure is handled after its production. How does the person assess the personal structure? Does he/she adhere to it or discard it? This splitting of the When into two parts, a lead-up and an assessment of the personal structure, appears in Sfard (2008) as *applicability condition* (*opening condition*) and *closing condition*. For the description of personal meaning, it is important to consider only those actions that the person has performed himself/herself.

The assessment of a personal structure is typically only possible once its consequences are clear. This typically manifests as manipulation but also, for instance, as a further personal structure. Overall, this article describes the personal meaning of a personal structure by dividing its chronology into its lead-up, personal structure, consequences and assessment. In this, the personal structure provides the How, while its lead-up and assessment provide the When, whereby the assessment is typically only possible in light of the consequences.

It is important to draw attention to four aspects here. Firstly, the above definition refers to a concept of personal meaning, distinct from the general concept of meaning. The distinction made here is based on the work of Godino and Batanero (1998) as well as Font, Godino and Gallardo (2013). These authors make a distinction between the individual and an *institution*, with the result that they are able to distinguish between a *personal* and an *institutional* meaning. The personal meaning referred to by these authors corresponds to the personal meaning defined here. For the definition of an institutional meaning, I refer to the sources quoted here. Secondly, the above definition of personal meaning does include an external aspect, in that the actions of other people are considered, of which the person has learned (c.f. second example above of an act of structuring). This assumes that mathematical activity is a *social* practice, as described in Godino and Batanero (1998); Radford (2001), or Wittgenstein (1978), for example. An essential feature of a social practice is that the use (and therefore the meaning) of terms is determined by rules, and all individuals involved in the practice judge to what extent the users of the terms are to follow these rules. Thirdly, the 'lead-up' (*Vorgeschichte*) defined in this article is not intended to be the same as the 'history' (*Geschichte*) used for example in the socio-cultural historic approach of Radford (2001). Rather, this 'lead-up' to a personal structure merely refers to possible patterns in the reactions of the person forming the personal structure. This means that the article focuses on an individual's personal method of approaching the rules which govern the practice of algebraic manipulation and which developed during its history, and not on the rules themselves. Fourthly, from the above definition of the personal meaning of an individual structure it follows that the individual, by structuring an expression, constructs for himself parts of the internal meaning of the expression. This is because, as immediately follows from the definition, actions which involve personal structures of the expression also involve the expression itself, and as a result these actions also contribute to the internal meaning of the expression.

1.2 Current study

The central question for this work is which personal structures are formed by subjects in terms and equations. As the personal structures which are formed fundamentally depend on the terms and equations that are provided, the central question is restricted to exactly one class of algebraic terms and equations, namely fractional expressions and equations. The research question can now be formulated more precisely with the help of the above concepts: Which parts are related to one another in fractional expressions and equations, and how? The answer is certainly dependent on whether an expert or a novice is interviewed. Therefore the differences between the personal structures that are formed are of interest. The empirical characteristics of the construct of personal structures will be considered. Are there different types of personal structures?

The inclusion of experts firstly allows a more accurate characterisation of experts and novices through the comparison of the two cohorts. Secondly, it opens up the opportunity to ask about the suitability of the personal structures. Especially since this study uses as its starting point the idea that every subject structures a given fractional expression individually, it is of interest to see to what extent the experts actually differ in their structuring. The picture painted by the literature is inconsistent. For example, with Chi, Feltovich and Glaser (1981) the experts' representations of the problem are remarkably homogeneous, as are the experts' strategies in solving a linear equation (Star & Newton, 2009). Conversely, Dowker (1992) writes about her investigation of mathematicians' estimation strategies: “The most striking result of this investigation was the number and variety of specific estimation strategies used by mathematicians” (p. 53). Each of these studies posed their questions differently and so their results are only comparable to a limited extent. Nevertheless, they do demonstrate that there is a need to examine to what extent two experts will form personal structures differently or in the same way.

2 Method

The method of investigation for the research question posed needs to fulfil two requirements. Firstly, it must facilitate conclusions regarding how experts and novices interpret strings of algebraic symbols (terms and equations). What do the experts and novices notice particularly in the fractional expressions? What acts of structuring are prompted by these observations? Which personal structures result from this? What are the consequences? How do they assess these? Secondly, the research question posed here is, to the best of my knowledge, unresearched. There are no existing data to fall back on, which would allow one to put forward hypotheses. Therefore the method must be appropriate for explorative research. Task interviews, filming the subjects whilst they are solving the tasks together, measuring eye movements, etc. would be possible. In consideration of the fact that the research question formulated above is being examined in detail for the first time, the choice fell on task interviews. This method is lean and allows a comprehensive exploration of the production of personal structures through targeted inquiry. However, a variation of methods is planned for future studies.

In an earlier study (Rüede, 2009), a sorting procedure (grouping equations by similarities in the approaches for solving them) found indications that, unlike experts, novices tended to establish relationships in fractional expressions that were oriented toward syntactical features or standard procedures. As described in Chi et al. (1981), it could be assumed that the method of task interviews would allow these results not merely to be replicated but in fact explored much more thoroughly. However, the task interviews conducted here do differ in one point from those in Chi et al. (1981). These authors repeatedly asked the subjects to think out loud (among other things). The study presented here, on the other hand, worked with introspection after—and not during—the performance of the task. The subjects were given time to look at the fractional expression presented to them before being interviewed. On the one hand, such (stimulated) recalls do have certain problems. No one can be sure whether the subjects accurately describe what they observed and considered during the task (Ericsson, 2006). On the other hand, thinking out loud while performing the task influences how it is performed. For the study presented here, however, the authenticity of the situation was key. This is why the subjects were given time to think the task over in order to then perform it without interruptions.

In order to ensure that the subjects examined the presented expression and spoke about the relationships they established in the expression, and did not simply manipulate it, firstly no aid of any kind was permitted in the interviews, as suggested in Chi et al. (1981), and secondly the tasks were deliberately constructed in order to necessitate precise visual inspection. Doing entirely without paper and pencil leads to a similar examination as with a sorting procedure. The interpretation of the provided expression in the form of the personal structure produced is important here, and it is these personal structures that were the subject of the investigation. Most novices and experts were not unfamiliar with doing without paper and pencil. Exercises in the most widely-used teaching aid for algebra tasks in Swiss gymnasiums (academic high schools) (Deller, Gebauer & Zinn, 2008) demand from time to time that students both simplify terms and solve equations such as $\frac{6}{x-6}-\frac{x}{x-6}=\frac{9}{9-x}$ (ibid., p. 110) or $\left(\frac{1}{x}-\frac{2}{3}\right)\left(\frac{1}{x}+7\right)=0$ (ibid., 111) mentally.

As mentioned above, expressions had to be constructed that would force careful examination, ones that could not simply be solved by blind manipulation. Therefore terms and equations were constructed that at first glance should activate a standard procedure that then proves inappropriate, and subsequently should prompt precise examination and the discovery of a trick. Secondly, it had to be possible to complete the tasks without paper and pencil. It should be possible not only for experts but also for novices who recognise the trick to see their way to the solution without paper and pencil. In this way, the constructed equations followed closely those used in a sorting procedure in Rüede (2009), for example.

This is why fractional expressions and equations were chosen. For this class of algebraic expressions and equations, there are standard procedures for solving fractional equations (for example removing the denominators through multiplication) and simplifying fractional terms (for example converting to a common denominator and adding). For fractional expressions it is easy to formulate tricky tasks that at first glance still evoke inappropriate standard procedures. Fractional expressions also allow vertical or even diagonal relationships as well as the horizontal ones. This diversity is important for researching the empirical formation of individual structures. In conclusion, it should be emphasised that the constructed tasks are not typical examples of how the simplification of fractions and solutions for fractional equations are typically taught, but are suitable for the exploration of personal structures.

2.1 Participants

12 novices and 12 experts were interviewed. The novices (6 female, 6 male) were aged 15 to 17 and were in their 9th or 10th school year. They had had an average of two to three years of algebra lessons and had covered subjects such as linear equations, factoring polynomials, the simplification of fractional expressions and solving of fractional equations. Algebra is valued highly at the upper secondary level in Switzerland, particularly at grammar schools preparing for university study. Grammar school pupils were therefore chosen as novices. They came from a total of three different grammar schools. Two of them had a curriculum focusing on the arts, four had a curriculum focusing on commercial law, and six had a focus on mathematics and

science. All of the novices had sat mathematics examinations on fractions and fractional equations. According to their teachers, four novices had received poor grades, four average grades and the remaining four excellent grades. Participation in the interview was voluntary.

The experts were 12 mathematics teachers at Swiss grammar schools (1 female, 11 male). They all have a university degree in mathematics and many years' teaching experience. Care was taken to ensure that all experts had an additional qualification such as teaching methodology (4), mathematics thesis (2), textbook author (1), practical teacher (5). Participation in the interview was voluntary.

2.2 Tasks

The tasks set were the following three expressions and four equations:

$$\frac{2x}{2x-2} - 2 \cdot \frac{x}{2x-2} \tag{expression1}$$

$$\frac{6x^3 + 9x^2}{4x+6} \tag{expression2}$$

$$\frac{6+12x}{6x+2} + \frac{12x+2}{6x+2} \tag{expression3}$$

$$\frac{20x^3 + 30x^2}{4x+6} = \frac{12x+18}{8x+12} \tag{equation1}$$

$$\left(\frac{1}{4} - \frac{x}{4}\right) \cdot \frac{x}{4x+1} + 4 \cdot \frac{x}{x-4} = \frac{4x}{x-4} \tag{equation2}$$

$$\frac{x+1}{3x+1} + \frac{2x}{3x+1} = x^2 \tag{equation3}$$

$$\frac{3x^2 - 18}{2x^2 - 12} = x - \frac{3x^3 - 18x}{x^2 - 6} \tag{equation4}$$

Only expression2, and perhaps expression3, is a typical textbook task.

2.3 Procedure

The novices were interviewed at their school and the experts at the University of Zurich. The interviews were task-based and semi-structured, and were conducted by the author.

At the beginning of each interview the subject was told how the interview would be structured and what other conditions applied, for instance that no aids, including paper and pencil, were allowed, and that there would be unlimited time to answer each question. After the explanation of what the interview would involve, the subject was given the first term and asked to think about how it could be simplified. Once the subject had made a suggestion, specific questions were posed in an attempt to reconstruct what considerations had gone through his or her head. Secondly—if necessary—the subject was asked to continue thinking about the task. This was followed by the second and then the third term. The same process was used with the equations, the only difference being that now the subject was asked how he/she reached a solution. The interviews generally lasted about 50 min in total.

In most cases, the fact that paper and pencil were not allowed meant that the subjects focused on forming personal structures. They avoided any major mental calculations and checked the personal structures they formed—if at all—by estimating their suitability. The subjects typically only named the end result if specifically asked to do so. The simple expression1 and equation3 were exceptions here. In expression1, for example, all of the experts gave the answer 0 without being prompted.

2.4 Data analysis

The analysis focuses on verbal data, as this was judged to be the most fruitful for exploring the personal structuring that took place. The evaluation was essentially carried out by means of content analysis (Mayring, 2003) and relied heavily on the techniques of didactic reconstruction (Gropengiesser, 2007). The first step was the transcription. In the second step, the transcripts were analysed in accordance with the essential personal structures that were produced. On average, two to three central personal structures were focused on per task and subject. In the third step, the transcripts were "broken up". For every central personal structure, information about the lead-up, the description of the personal structure, its consequences and its assessment was sought. This then led to a manipulation, arranging and editing of the transcripts. In the editing, attention was paid to authentic wording. The descriptions obtained in this way formed the basis of the interpretation. This interpretation consisted of inductively generating categories of key personal structures, which produced four categories.

3 Results

The results are presented in two parts. The first part shows how each expert structured the given fractional expressions in a different way. Four categories of personal structure are then described, namely syntactic, operational, first-order and second-order.

3.1 12 experts cancelled in 12 different ways

Every expert structured in a different way. This diversity is best shown by looking at expression2. In this term, the experts were expected to be unanimous about the procedure. In fact all the experts cancelled, but all of them did it differently. Tables 1, 2 and 3 describe the personal structures the experts produced during the first period of thinking time. Due to reasons of space, only the central personal structure is focused upon.

The personal structures in Table 1 emphasised vertical lines of vision. In this aspect they are similar, but they differ in the detail. Expert5 and Expert8 related in each case the coefficients and the powers to one another vertically. Expert11 related almost the entire numerator vertically to the entire denominator. Other experts looked horizontally, as shown in Table 2.

The four experts in Table 2 proceeded in textbook fashion. They factorised. This algorithm governed their structuring. They related the coefficients to each other horizontally

Table 1 Three experts' vertical personal structures in task (2). Actual quotes from the interviews are given in quotation marks

Expert5: Summands of the numerator are $1.5x^2$ times as large as those of the denominator	
Lead-up	I look "naturally at the fractions, at what I can cancel".
Personal structure	"It's this proportionality of the numbers, 6, 9, 4, 6, where I see that these coefficients in the numerator are 1.5 times as big as they are in the denominator."
Consequences	"And then I checked whether the [...] numerator is bigger than the denominator by a factor of x^2."
Assessment	"At first glance, there's an initial suspicion that it could make $\frac{3}{2}x$. Now I'll check [...]"
Expert8: Coefficients of the denominator are $\frac{2}{3}$ times those of the numerator.	
Lead-up	I look to see "whether I can cancel out a common factor".
Personal structure	I "noticed that that is $\frac{2}{3}$ of that one and that is $\frac{2}{3}$ of that one" (*points to the 4 on the bottom, with 6 on top and at the 6 on the bottom with the 9 on top*)
Consequences	"Then something just had to be able to be cancelled, at some point from the numbers…"
Assessment	" … and if the powers are also right, then it would have to go"
Expert11: I compare the top with the bottom to find a common factor.	
Lead-up	"I want to find the same terms in them [...] You expect something like that in this type of term."
Personal structure	I placed x^2 outside the brackets "and then I see that I've got 6 and 9 and 4 and 6. I look for similarities in the $6x+9$ and the $4x+6$ [...] I always looked at both of them."
Consequences	I "tended to see what was in the bracket"
Assessment	"I have to check it two or three more times to see whether this factor is there."

in each case by identifying common factors. The personal structures differed in their lead-up and in the order in which the actions took place. Some first related the coefficients in the denominator to one another, while others started with the numerator. A few first identified common factors of the numbers, others those of the powers. Overall, these horizontal personal structures are also similar, but differ in the detail. The vertical aspects are marginal, and limited to dividing expression2 into above and below, and comparing the factors outside the brackets above the line with the ones below.

For the rest of the experts, horizontal *and* vertical personal structures were fundamental (Table 3).

Expert3, Expert2, Expert10 and Expert1 related the coefficients to each other horizontally by forming connections that they then related to each other vertically. Expert3 spoke explicitly of the factor of 1.5, Expert2 of proportionality. On the other hand, some first related terms in the denominator to each other, while others related parts in the numerator to each other. Differences could also be discerned in the lead-ups and the assessments. Expert12 tended toward personal structures as in Table 2. However, in this, he used vertical relationships to a greater degree than the experts in Table 2.

There is evidently not *one* personal structure of a fractional expression, but *many* appropriate personal structures of a fractional expression. Something that the experts had in common is that they produced relationships between parts in order to establish suppositions about a common linear factor. This is a key characteristic of the personal structures of the experts (apart from Expert9, who did not make suppositions, but

Table 2 Four experts' horizontal personal structures in task (2). Actual quotes from the interviews are given in quotation marks

Expert4: There are no common numerical factors, so I factorised in the denominator and then in the numerator	
Lead-up	"So, I thought there must be a simplification, because the questions are obviously set up that way […] First I wanted to cancel a natural number and then I saw that it wouldn't work. Then the second approach was next, […] which was, what other factors are there. "
Personal structure	"So definitely take the 2 out of the brackets on the bottom, then you've got $2x+3$, and on the top, the 3 or $3x^2$ […]"
Consequences	"[…] then you've got $2x+3$"
Assessment	"And now I've got to say what's left."
Expert6: Separate the denominator and then the numerator completely into factors.	
Lead-up	"I thought that if you can simplify it at all then only through separating it into factors"
Personal structure	In the denominator "I saw that […] you can only separate the factor 2". "I would take apart [the numerator] […], I think I saw the 3 first", then the x^2.
Consequences	"That then leads to this factor."
Assessment	"Due to the posing of the task […], I assume that a […] linear factor can be separated"
Expert7: First I factorise in the numerator, then in the denominator.	
Lead-up	"The task is probably put together so that something can be cancelled, that a lot can be cancelled"
Personal structure	"So first I saw the many x's in the numerator and then I thought that you can factor out x^2 […] in the numbers […] you can see that the 3 on the top and the 2 on the bottom can be factored out. "
Consequences	"Now you just have to work it out."
Assessment	"I thought that the same could probably be there, that the task is set up like that."
Expert9: Routine option F. I factorise in the denominator, then in the numerator.	
Lead-up	"So as it is put together, an amount on top and an amount at the bottom, that's routine option F"
Personal structure	"I look at what I could bring out here (*indicates the denominator*) and here, I think I'll bring out the 2, that's doubled, then I look at what I could bring out here (*indicates numerator*), I see x^2, $3x^2$, the 3 and the x^2 is doubled."
Consequences	"Then I see that the $2x+3$ is in there"
Assessment	I looked at "the denominator first, because it's easier". "This is how I teach the students to do it"

calculated immediately). In doing so, half the experts related parts to one another in such a way that they were able to justify their supposition on the basis of proportionalities (Expert5, Expert8, Expert3, Expert2, Expert10, Expert1). The important fact here is that in order to do this, parts must have been at least partly related to one another on a vertical level. The remainder of the experts referred to context and the type of exercise. They only related terms to one another in order to test whether something could be factored out.

In order to underscore this feature of the experts' structuring, I shall take a look at the novices' structuring. The novices tended to relate parts to each other horizontally, and therefore rarely formed suppositions, instead simply carrying out the manipulations that

Table 3 Five experts' horizontal and vertical personal structures in task (2). Actual quotes from the interviews are given in quotation marks

Expert3: Coefficients of the two summands are in each case 1.5 times that of the first	
Lead-up	I look at "whether I see common parts in the numerator and the denominator."
Personal structure	I see "the numbers already, the coefficients 6 and 9, 4 and 6, I can see already that the second one is 1.5 times the first one there…"
Consequences	"… then it has to work"
Assessment	"Then I should consider what I can take out as a factor now."
Expert2: The coefficients 4 and 6 in the denominator are proportional to 6 and 9 in the numerator.	
Lead-up	"I always look for identical factors in the numerator and the denominator. And now, because there's nothing to do with the numbers, I start on the factorising."
Personal structure	Then I saw "the proportionality between 4 and 6 (*indicates the denominator*) and 6 and 9 (*indicates the numerator*)"
Consequences	"I suspected that there was an identical factor in there somewhere"
Assessment	"Yes, nice"
Expert10: The coefficients 6 and 9 in the numerator are proportional to 4 and 6 in the denominator.	
Lead-up	"It can only be about cancelling, so, time to factorise"
Personal structure	I quickly noticed the relation "of 6, 9, 4, 6"
Consequences	"In principle" you can cancel
Assessment	"But actually, I had to calculate again afterwards, and I wasn't sure"
Expert1: $2x+3$ at the bottom and look at whether that fits with the top	
Lead-up	"Actually, it's clear that you probably have to place it outside the brackets, factor out"
Personal structure	"There are two thoughts. First x^3, x^2, so there you can factor out the square, and then it's like a check as to whether that also works with the figures […] 6 and 9 and 4 and 6".
Consequences	"You need to factorise there." I see straight away "the $2(2x+3)$" and look to see "whether it then fits".
Assessment	"If you're lucky", you can "cancel […] $2x+3$ on the top and the bottom"
Expert12: I can probably place a factor outside the brackets in the denominator, which appears again on top.	
Lead-up	I thought "like a student". "You can probably simplify it or something. Then let's see if we can factor out something in the denominator so that it appears on the top."
Personal structure	"In the denominator, you can take 2 outside the brackets, then you get $2x+3$." Then I test for "what I need to take out of the numerator so that $2x+3$ is left".
Consequences	"I take x^2 out of the brackets, and then $6x+9$ is still there, but I need $2x+3$, so I take out 3 as well".
Assessment	To be safe, I multiplied them both out again.

resulted from the relationships. For example, not one of the novices related parts to each other vertically in the way that the three experts did in Table 1. Seven novices related the parts horizontally in each case, such as Novice 3 for example.

Novice3: First, always get rid of the x^3, the x to the power of something	
Lead-up	I noticed the x^3, "so that's just, I don't know, what my mother would say. First, always get rid of the x^3, the x to the power of something […] and then I just factored out."
Personal structure	"[It has] an x^2 there, and the x^3 […] so I factor out x^2"

Consequences	"And then that gives x^2 times $6x+9$ […] then I can also factor out the $6x+9$, so factor out the 3, and then that gives $3x^2$ times $2x+3$ and below I factor out a 2 and that gives $2x+3$."
Assessment	"And then I can cancel out the brackets"

Novice3 produced relationships in order to manipulate the expression and not in order to establish (or verify) suppositions as the experts did.

Three novices related the parts both horizontally and vertically, such as Novice6 for example.

Novice6: It seemed clear to me from the beginning

Lead-up	"I'm not allowed to cancel here, of course […] because there are two additions."
Personal structure	Then I "just saw right away 6 and 9 and 4 and 6. They each have common factors." And here I have "x^3, x^2, x and no x."
Consequences	"Then I already suspected that it would produce the same brackets."
Assessment	"It seemed clear to me from the beginning."

Novice6 is one of two novices overall who formed personal structures in the same way as the experts, that is, he attached suppositions to his personal structures. Novice6 used proportions to mathematically justify the existence of a common linear factor.

The remaining two novices (Novice4 and Novice11) were the only ones not to discover during the thinking time in the interview that one could cancel the linear factor $2x+3$, as the following passages show.

Novice4: What's in front of the brackets can't be cancelled

Lead-up	"Whenever I see common factors, common variables in a fraction, I try to factor them out."
Personal structure	"So the x^3 and the x^2. First of all the x^2 disappears behind, and the x^3 somehow becomes x. Then […] 6 and 9 […] so 3 and 2 below."
Consequences	"But these can't cancel each other out."
Assessment	This was a "dead end. I just looked at what was in front of the brackets. And then didn't think any more about what was inside the brackets."

Clearly Novice4 related the parts to each other in order to cancel the coefficients. The focus was never on the brackets, but only on what came before the brackets.

3.2 Four categories of personal structures

As expected, the most informative personal structures for the creation of categories were the ones where subjects produced relationships that allowed them to carry out manipulations which were not oriented towards algorithms. Situations involving exploration and discovery were particularly helpful, during the evaluation, in the creation of four categories of personal structures. They will be demonstrated using Eq. 2 as an example.

$$\left(\frac{1}{4}-\frac{x}{4}\right)\cdot\frac{x}{4x+1}+4\cdot\frac{x}{x-4}=\frac{4x}{x-4}$$

It is significant for Eq. 2 that it is easy to solve as soon as the equality of the two summands $4\cdot\frac{x}{x-4}$ and $\frac{4x}{x-4}$ is recognised. However, that was a challenge for some of the subjects.

3.2.1 Syntactic structuring (Novice11, female)

A description of the first central personal structure of Novice11 follows.

Lead-up	I always look for what's similar first "because that makes it easier". My strategy is "crossing things out, but that's always so awkward."
Personal structure	"First I looked at the brackets. Then $\frac{1}{4}-\frac{x}{4}$. I wondered whether I could've crossed out the 4."
Consequences	If I'd crossed out the two 4's, it would have been $1-x$.
Assessment	"But it doesn't work, because it's divided into quarters and the x can be any number. And then you'd have so many quarters and not so many whole numbers. And that'd be wrong. I thought, no, you can't do that, I'll leave it as it is."

Novice11 then related the two denominators $x-4$ to one another. The equality of these denominators then gave her the idea of determining the common denominator and multiplying the equation by it. She suggested this manipulation after the thinking time.

Categorisation:

In every task, Novice11 first formed relationships between things that looked the same or similar, with the intention of simplification. She treated the algebraic expressions similarly to how an illiterate person treats lettering. To an illiterate person, graphical similarities in lettering are important. Analogously to this, Novice11 focused on what was syntactically similar in the algebraic expressions provided. Therefore I will refer to personal structures such as this as *syntactic*. In this approach, similar parts are related to one another. How the parts were related to one another was suggested by the operation sign in each case. In the above instance, there was a minus sign in between the two '4' denominators. Therefore Novice11 related the two denominators to each other so that these cancel out due to their equality in combination with the minus sign. $1-x$ is visually more straightforward than $\frac{1}{4}-\frac{x}{4}$.

The passage cited in the assessment clearly documents Novice11's struggle to find the correct rules for the combination of algebraic symbols. She related the parts to one another in order to establish which operations were behind it all, or, in the words of Kieran (1989), what type of surface structure was involved. If Novice11 was certain of the correctness of the consequences of her personal structure, she stood by it; otherwise she discarded it.

3.2.2 Operational structuring (Novice3, female)

Novice3 recognised the type of equation "$A+b\cdot\frac{B}{C}=\frac{bB}{C}$" because this type had been discussed in her class. She followed the corresponding procedure, namely multiplying 4 by $\frac{x}{x-4}$ and then removing both fractions by subtraction. She erroneously multiplied the top and bottom by 4 and was therefore not able to use the procedure. After that she related the parts to each other as follows.

Lead-up	So I wasn't able to "simply get rid of the fractions". "Then I came […] to the brackets". I wanted to try "removing the brackets". Because "I didn't know what else to do with the numbers in them. Because I didn't really have anything that could be got rid of straight away, or that anything that you could somehow find straight away".
Personal structure	"So it's easier to do that (*indicates* $\frac{1}{4}$) times that (*indicates* $\frac{x}{4x+1}$) and that (*indicates* $\frac{x}{4}$) times that (*indicates* $\frac{x}{4x+1}$)."
Consequences	"I just know that afterwards there'll be $16x$ somehow". The bracket is then "removed"
Assessment	"that the bracket is removed".

Categorisation:

After Novice3 was not able to execute her procedure, she did not know which parts she should then relate to one another. She looked for something she could do. She followed the hierarchy of operations, and therefore looked at what could be manipulated to begin with. First, the brackets needed removing. But because she was unable to combine the two fractions into a single fraction, she saw only the possibility of multiplication. Her personal structure followed the distributive law $(a+b)c = ac + bc$. She related $\frac{1}{4}$ and likewise $\frac{x}{4}$ to $\frac{x}{4x+1}$ in a multiplicative way. That is *operational* structuring. Parts that belong together visually, such as fractions, bracket terms and numbers, are related to one another, with the goal of changing the expression. In the case of Novice3, this goal consisted of the 'removal' of the brackets: Novice3 anticipated that $16x$ would be somewhere in the denominator and, above all, that the brackets would no longer exist.

This operational structuring was not geared towards solving but towards changing the equation. Novice3 assessed whether this would cause the brackets to no longer appear and not whether the manipulation leads to the solution of the equation. Novice3 just wanted to do something. This is precisely the goal of operational structuring. In this context, Sfard (2008) speaks of "actions that, unlike explorations, produce or change the objects" (p. 237).

3.2.3 First-order structuring (Novice2, female)

Novice2 first related the denominators to one another in order to determine the common denominator. Due to the diversity of the denominators, she discarded this personal structure as it was too complicated. She then considered whether it was possible to approach the equation differently.

Lead-up	"Then I looked at whether you can do it differently. […] I also took the 4 up again to the x […] and then saw that they're the same".
Personal structure	$4 \cdot \frac{x}{x-4}$ on the left and $\frac{4x}{x-4}$ on the right side are the same
Consequences	Then on the left side "an addition appeared […] and here [on the right side] it'd also be like 0 plus, and then you could take it away from both sides".
Assessment	"Then you could easily multiply it by the denominator and you'd simply have $x(1-x)=0$" left.

Categorisation:

This personal structure combines two views. Novice2 treated the term $4 \cdot \frac{x}{x-4}$ firstly as multiplication. The internal element, i.e. the multiplication, was important. She multiplied. The multiplication became a product. She tried to relate this to other parts. She related the part on the left of the equals sign to the part on the right of it, and recognised that they were the same. Due to this comparison, she increasingly saw $4 \cdot \frac{x}{x-4}$ from an external viewpoint. What is acting on this product? A plus sign; $4 \cdot \frac{x}{x-4}$ is also a summand. That is formulated in the above consequences. The assessment can be extrapolated indirectly. Novice2 uses the words “easily” and “you’d simply have”. She recognised the profit of the personal structure for the solution of the equation.

I refer to this type of personal structure as *first-order*. It expresses a reinterpretation. An additional meaning is attributed to the same part of the expression. In the above equation, Novice2 first treated $4 \cdot \frac{x}{x-4}$ as a multiplication, then as a product, and finally as a summand. The interpretation changed from an internal to an external one. Though the internal feature of “multiplication” was important at first, it was ultimately the external feature of the “summand” which was important.

3.2.4 Second-order structuring (Expert12, male)

The first personal structure formed by Expert12 led him straight to the solution.

Lead-up	“Actually I’m adding two fractions there, which gives a third, and then it always gets difficult when I have to solve it. But if I still had a product, which would be 0, then I could do something. Then I try to get rid of this one, $4 \cdot \frac{x}{x-4}$ ”.
Personal structure	“Then I see this (*indicates* $4 \cdot \frac{x}{x-4}$) comes up again here (*indicates the right-hand side*) exactly the same”.
Consequences	“So we subtract it”.
Assessment	“I probably didn’t think about $A \cdot B$, about the letters. But in the sense that I’ve got a product there, then, yes. I think that more than anything else it is the plus sign that is annoying. […] And then I tried it. It worked”.

Categorisation:

What Novice2 had to work hard for, Expert12 did from the start: he looked at the term from the outside towards the inside. He immediately treated the equation as the addition of two fractions, which produce a third.[1] This personal structure should be viewed against the backdrop of looking for a common denominator. Expert12 thought that this would be a lengthy procedure and therefore looked for alternatives. He continued to work from the outside inwards and recognised the multiplication in the first summand. He considered the possibility of a special case. Perhaps an equation in the form of $A \cdot B=0$ could be produced somehow. For this reason he then related the two summands on the left and right of the equals sign to one another. In the assessment, he indirectly defines a criterion for standing by this personal structure, namely, if the two terms of the sum are the same and in fact an equation of the type $A \cdot B=0$ results. This means that Expert12 judges his personal structure based on whether it leads to an elimination of the “annoying” plus sign.

[1] This is also second-order structuring. But it is not what is being discussed at present.

I refer to such personal structures as *second-order*. They are based on the subject already being familiar with the diverse interpretations of individual parts and their corresponding procedures, and no longer having to learn these. He therefore forms personal structures for the purposes of deciding which procedure is appropriate to use and when. His production of the personal structures is an exploration of the classification of the equation. I speak of second-order and not first-order personal structures for this reason. Chi et al. (1981) also speak very similarly of *second-order features*. According to these authors, experts identify second-order features when reading physics exercises. Features like this are not explicitly stated in the text of the physics exercises; rather, explicit passages are *read as* this type of feature. For the experts, these second-order features decide which interpretation is chosen.

3.3 Four categories of forming personal structures

The above four examples show that structuring empirically separates out into the activities of visual simplification, of change, of reinterpretation and of exploration of a classification (Table 4).

3.3.1 Syntactic structures

Syntactically similar elements give rise to syntactic structuring. Syntactic structuring separates the two-dimensional arrangements of the symbols into what is the same and what is dissimilar. The goal is making the expression more visually straightforward. Whether the personal structures are accepted or rejected depends on the degree of certainty as to whether a correct manipulation will result.

In the above example above of equation2, Novice11 related the same denominators to one another in order to cross them out due to the minus sign. In equation1, in the fraction $\frac{12x+18}{8x+12}$ she related $12x$ to 12 because that is “simpler”, and 18 to $8x$, because these then would be left over. Both times, she asked herself whether it was possible to perform division.

3.3.2 Operational structuring

The need to change an expression leads to operational structuring The subject simply wants to do something, regardless of whether or not the personal structure contributes to completion of the task. Subjects correspondingly retained an operational personal structure depending on whether a manipulation was possible—and not on whether a manipulation was profitable.

Two subcategories of operational structuring were isolated. Either the personal structure followed a given operation sign or the expression was just manipulated in whatever way was possible. An example for the first case is the personal structure that Novice2 formed in the above example of equation2. The hierarchy of operations governed this. Novice2 multiplied

Table 4 Four categories of structuring

Personal structure	Syntactical	Operational	First-order	Second-order
Action	Visual simplification of the expression	Changing the expression	Generating new interpretations of the expression	Exploring classifications of the expression

out. An example for the second case is Novice10's personal structure in equation1, $\frac{20x^3+30x^2}{4x+6}=\frac{12x+18}{8x+12}$. When asked where he looked first, he said, "So I looked at what I had to do with these denominators to make them equal". Novice10 multiplied the denominators away in all tasks except for expression2. In equation1, he searched the equation for data to use in this procedure. He "saw 12 and 6, and 4 and 8 and […] 4 is half of 8, and 6 is half of 12". This is what his operational personal structures looked like in detail. Therefore he expanded the left side by 2. His final result was $18+12x+30x^2+20x^3$. He said later that he must have seen the equals sign "as a plus sign somehow". Evidently, Novice10 simply performed manipulation in whatever way seemed possible to him. The structuring served solely to change equation1.

3.3.3 First-order structuring

The starting point for first-order structuring is an initial interpretation of the expression. The goal is the generation of a new interpretation. It is therefore reinterpreted. This is accompanied by a new division of the two-dimensional arrangement of the symbols. Whether the new personal structure is retained is decided by its value for the solution of the task.

In the above example, Novice2 reinterpreted the left side of equation2. She first treated $4\cdot\frac{x}{x-4}$ as a multiplication and ultimately as a summand. She achieved this reinterpretation by working from the inside outwards. Movements of this kind from the inside to the outside could often be reconstructed in the tasks provided here. However, the reverse was also observed. For example, for Novice7 on the left side of equation3, $\frac{x+1}{3x+1}+\frac{2x}{3x+1}=x^2$, the external feature of fractional terms was important. She removed the denominator by multiplication. Only when she was asked for a second variant did she produce an additive personal structure on the left side between the two fractions. She realised that she "could simply lengthen the fraction line". She subsequently also related the two numerators to one another additively. In this way, she reinterpreted $x+1$ as a summand (of one numerator) and no longer just as part of a fraction.

3.3.4 Second-order structuring

Each expert was aware of the ambiguity of expressions and needed decision-making criteria as to which interpretation was appropriate and when. Second-order personal structures were used to this end. The goal is determining which interpretation of the expression to choose, that is, its classification. Whether the personal structure is retained or not is decided by the appropriateness of the interpretation associated with it.

In the above example, in equation2, Expert12 related $4\cdot\frac{x}{x-4}$ and $\frac{4x}{x-4}$ to one another in order to decide whether equation2 is of the form $A\cdot B=0$. Another example was Expert5. In equation1, $\frac{20x^3+30x^2}{4x+6}=\frac{12x+18}{8x+12}$, he first related the parts to one another in such a way as to gain information about multiplying away the denominator. He discarded this approach and took from it the supposition that the two sides should be simplified individually. To this end, he produced horizontal relationships that he related to one another vertically: "20 and 30 in the first fraction for example, have the same proportionality as 4 and 6. Or even x^3 to x^2 as x to 1, or 12 to 18 as 8 to 12". He recognised that the relationships were the same.

These personal structures were typical of the experts. They formed a supposition about a possible approach, and then structured in such a way as to confirm or deny this supposition. In this sense, they explored the possible classifications of the expression; they carried out *explorations* (Sfard, 2008).

4 Discussion

This article focuses on individual ways of handling algebraic expressions. Therefore it defines the 'structuring' of an expression as the act of forming relationships between its parts. On one hand, this definition allows us to contrast the concept of 'structure', which is firmly embedded in mathematical theory, against a concept of 'personal structure', oriented towards mathematical practice. On the other hand, as soon as structuring is defined as an action, it is possible for it to be observed. The study presented here makes use of this empirical accessibility. The different personal structures formed by experts and novices were examined. Firstly, the empirical results show that each expert formed his/her own personal structures. Even where mathematical theory, for example with the expression $\frac{6x^3+9x^2}{4x+6}$, would recommend one exact structure such as $\frac{A}{B}$, it was found that each expert formed his/her own personal structure in the course of simplifying the fraction. There is not *a single* personal structure. What the experts' personal structures did have in common, however, is that they served the purpose of setting up and verifying suppositions. This was very rarely observed when it came to the novices. This difference between the experts and the novices is expressed in the four categories of personal structures which were identified. Syntactic structuring makes an expression visually more simple, while operational structuring changes it, first-order structuring reinterprets it and second-order structuring explores possible classifications of it. For the most part, the novices formed personal structures corresponding to the first three categories, with the experts' personal structures corresponding to the fourth. First-order personal structures are of central importance in the teaching of algebra. This is firstly because they prove that, while productive interpretations of expressions are not obvious, students are still absolutely capable of discovering them independently. Secondly, reinterpretations seem to be a prerequisite for being able to treat algebraic terms as objects; this is because a reinterpretatation attributes two different interpretations to the same string of symbols. According to Brandom (1994, p. 424), this is a prerequisite for any kind of objectification: "To be an object is to be something that can be referred to in different ways." Therefore a person can only objectify those terms that he/she is able to treat (and in particular, structure) in two different ways.

At first glance, the observation that both experts and novices form personal structures in different ways could be seen as contradicting Star and Newton (2009), where the experts more or less agreed about the best strategy for solving an equation. However, firstly, the tasks used by Star and Newton allow fewer different solution strategies than the tasks here. Secondly, the experts in Tables 1, 2 and 3 agreed about what the best strategy was. All of them indicated that cancelling should be performed on expression 2. Using the concepts of Hoch and Dreyfus (2004, 2010), it can be said that all of them saw the same structure, namely $\frac{A}{B}$. However, this article is interested in the interpretations of $\frac{A}{B}$ and "cancelling". It describes these interpretations by reconstructing which personal structures were produced under this name. Evidently, they are individual.

The question remains open as to whether the four categories (Table 4) are developmental levels, levels of ability, or different types of personal structures. The study introduced in this article is purely explorative. It serves to provide an empirical description of personal structures, and not to record abilities and developmental levels. However, because this question will be the object of subsequent studies, a few considerations regarding this will be mentioned at this point. The evaluation of the interviews gave rise to the hypothesis that, in Table 4, the level of expertise increases from left to right. In this respect, it could be assumed that the four categories relate to abilities and perhaps also to the development of structuring. Sfard (2008) argues that approaches generally develop from *deeds* to *rituals* to *explorations*. Sfard (1991) also states that interpretations of algebraic expressions develop from *operational* to *structural*. And in fact, deeds, rituals or operational interpretations are towards the left in Table 4. To the right are the structural interpretations. Second-order structuring is explorative. Expert12 first established a theory that the terms on both sides of the equals signs cancel each other out, and checked this. In this respect, a development in Table 4 from left to right could be assumed. There is currently, however, no model of how this development could take place, unless I employ Sfard's arguments (1991, 2008). In addition, developments in descriptive terms are always fundamentally dependent on concrete teaching processes. Therefore, the hypothesis should be formulated more cautiously, if at all: In Table 4 levels of ability are given from left to right. Even if this is correct, a concrete approach to teaching ought to be included in the formulation of the hypothesis, because current teaching in algebra and also arithmetic follows exactly this direction from the operational to the structural.

The concept of structuring as the act of forming relationships between parts, as introduced in this article, suggests that structuring through *relational thinking* (Carpenter, Franke & Levi, 2003) should be encouraged. Up to this point, it is predominantly primary school students who have been led towards relational thinking. Carpenter, Franke and Levi suggest two task formats to encourage such relational thinking. In one task format, the children have to find the number in the box in expressions such as $67+85=\square+84$. The goal is that the children do not calculate, but relate parts to other parts instead, such as that 85 is 1 bigger than 84 and so the number in the box must be 1 bigger than 67. In the other task format, the children need to investigate the correctness of some statements and justify their answers. Carpenter, Franke and Levi suggest statements such as $6\cdot 7=5\cdot 7+7$. This is also a case of mathematical rules, and not calculating arithmetical expressions. The two task formats can be transferred to algebraic structuring. For fractions and fractional equations, on the one hand tasks such as

$$\frac{2x}{x^2+1}+\ldots=\frac{3x}{x^2+1} \quad or \quad \frac{3x^2+1}{6x^2+2}=\frac{\ldots}{4x^4+4}$$

suggest themselves.

Here, the gaps need to be filled in to make the equations universally valid. Alternatively, statements such as

$$\frac{4-8x}{2-8x}=2 \quad or \quad \frac{x^2}{1+x}=(2+x)\cdot\frac{x}{1+x}-2\cdot\frac{x}{1+x}$$

can be investigated for their universal validity. In this task, the students should be asked to look at the expressions, to compare individual parts with one another, to perform as few manipulations as possible, and to justify their decisions. The idea is to pose this type of task as an accompaniment to the lesson and not to process dozens of them in several lessons. In all events, they should be used to prompt conversation in order to make people's personal structures explicit, to compare them with those of other classmates and therefore to contrast the appropriate and inappropriate personal structures. Through this type of conversation in class, the students should realise that algebraic expressions are not to be understood as an invitation for them to execute learned procedures, but as a challenge to develop their own approach to those expressions. It is extremely helpful for students to learn by example what personal structures other people produce, as well as how they use them. It is through such comparisons that the students learn to reinterpret expressions—according to the hypothesis—and therefore to generate first-order personal structures.

After all, the appropriate way to treat an algebraic expression is a matter for negotiation. Due to this type of discourse, the students naturally develop the need for suitable vocabulary. They recognise the value of technical vocabulary for describing their own and other people's personal structures. Language becomes important. Only a shared technical vocabulary (Hoch & Dreyfus, 2010; Kirshner & Awtry, 2004) can enable mutual understanding. This is because students and teachers are only able to make connections between their different personal structures of an algebraic expression in a discursive way. The technical language is the medium with which to make explicit at what point one is relating parts of an expression to other parts, and how. Teachers need to achieve clarity on the personal structures the students are creating, for example what they mean when they are talking about cancelling in expression2. As well as this, teachers can direct the students' personal structures with the aid of language. The precise formulations necessary for this are only useful in the framework of a common technical vocabulary. Teachers and students can only take part in a subject-specific discourse on the appropriateness of personal structures with the aid of this vocabulary.

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